

The Bicomplex Quantum Harmonic Oscillator

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Abstract

The problem of the quantum harmonic oscillator is investigated in the framework of bicomplex numbers, which are pairs of complex numbers making up a commutative ring with zero divisors. Starting with the commutator of the bicomplex position and momentum operators, and adapting the algebraic treatment of the standard quantum harmonic oscillator, we find eigenvalues and eigenkets of the bicomplex harmonic oscillator Hamiltonian. We construct an infinite-dimensional bicomplex module from these eigenkets. Turning next to the differential equation approach, we obtain coordinate-basis eigenfunctions of the bicomplex harmonic oscillator Hamiltonian in terms of hyperbolic Hermite polynomials.

1 Introduction

The mathematical structure of quantum mechanics consists in Hilbert spaces defined over the field of complex numbers [1]. This structure has been extremely successful in explaining vast amounts of experimental data pertaining largely, but not exclusively, to the world of molecular, atomic and subatomic phenomena.

That success has led a number of investigators, over many decades, to look for general principles or arguments that would lead quite inescapably to the complex Hilbert space structure. It has been argued [2, 3], for instance, that the formulation of an uncertainty principle, heavily motivated by experiment, implies that a real

Hilbert space can in fact be endowed with a complex structure. The proof, however, involves a number of additional hypotheses that may not be so directly connected with experiment. In fact Reichenbach [4] has shown that a theory is not straightforwardly deduced from experiments, but rather arrived at by a process involving a good deal of instinctive inferences. The justification of the theory lies in its ability to explain known experimental results and to predict new ones. More recently, some of the efforts to derive the complex Hilbert space structure have focused on information-theoretic principles [5, 6]. The general principles assumed at the outset are no doubt attractive, but yet open to questioning. The upshot is that there is no compelling argument restricting the number system on which quantum mechanics is built to the field of complex numbers. A possible extension of quantum mechanics to the field of quaternions was pointed out long ago by Birkhoff and von Neumann [7], and it has since been developed substantially [8, 9].

The fields of real (\mathbb{R}) and complex (\mathbb{C}) numbers, together with the (non-commutative) field of quaternions (\mathbb{H}), share two properties thought to be very important for building a quantum mechanics. Firstly, they are the only associative division algebras over the reals [10]. A *division algebra* is one that has no zero divisors, that is, no non-zero elements w and w' such that $ww' = 0$. Secondly, they are the only associative absolute valued algebras with unit over the reals [11]. An *absolute valued algebra* is one that has a mapping $N(w)$ into \mathbb{R} that satisfies

- i. $N(0) = 0$;
- ii. $N(w) > 0$ if $w \neq 0$;
- iii. $N(aw) = |a|N(w)$ if $a \in \mathbb{R}$;
- iv. $N(w_1 + w_2) \leq N(w_1) + N(w_2)$;
- v. $N(w_1w_2) = N(w_1)N(w_2)$.

Property (v), in particular, is widely believed crucial to represent quantum-mechanical probabilities and the correspondence principle with classical mechanics.

Yet several investigations have been carried out on structures sharing some characteristics of quantum mechanics and based on number systems that are neither division nor absolute valued algebras [12, 13]. Of these number systems the ring \mathbb{T} of bicomplex numbers is among the simplest. It has already been shown [14] that structures analogous to bras, kets and Hermitian operators can be defined in finite-dimensional modules over \mathbb{T} .

In this paper we intend to pursue that investigation further by extending to bicomplex numbers the problem of the quantum harmonic oscillator. The harmonic oscillator is one of the simplest and, at the same time, one of the most important systems of quantum mechanics, involving as it is an infinite-dimensional vector space.

In section 2 we review the main properties of bicomplex numbers that we will use, together with the notions of module, scalar product and linear operator. Section 3 is devoted to the determination of eigenvalues and eigenkets of the bicomplex quantum harmonic oscillator Hamiltonian, along lines very similar to the algebraic treatment of the usual quantum-mechanical problem. To our knowledge, this is the first time that such eigenvalues and eigenkets are obtained with a number system larger than \mathbb{C} . An infinite-dimensional module over \mathbb{T} is explicitly constructed with eigenkets as basis. Section 4 develops the coordinate-basis eigenfunctions associated with the eigenkets obtained. This leads to a straightforward and rather elegant generalization of the usual Hermite polynomials as hyperbolic functions of a real variable. Section 5 connects with standard quantum mechanics and opens up on new problems.

2 Bicomplex numbers and modules

This section summarizes basic properties of bicomplex numbers and finite-dimensional modules defined over them. The notions of scalar product and linear operators are also introduced. Proofs and additional material can be found in [14, 15, 16, 17].

2.1 Algebraic properties of bicomplex numbers

The set \mathbb{T} of *bicomplex numbers* is defined as

$$\mathbb{T} := \{w = w_e + w_{i_1}i_1 + w_{i_2}i_2 + w_jj \mid w_e, w_{i_1}, w_{i_2}, w_j \in \mathbb{R}\}, \quad (2.1)$$

where i_1 , i_2 and j are imaginary and hyperbolic units such that $i_1^2 = -1 = i_2^2$ and $j^2 = 1$. The product of units is commutative and defined as

$$i_1i_2 = j, \quad i_1j = -i_2, \quad i_2j = -i_1. \quad (2.2)$$

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set \mathbb{T} makes up a commutative ring.

Three important subsets of \mathbb{T} can be specified as

$$\mathbb{C}(i_k) := \{x + yi_k \mid x, y \in \mathbb{R}\}, \quad k = 1, 2; \quad (2.3)$$

$$\mathbb{D} := \{x + yj \mid x, y \in \mathbb{R}\}. \quad (2.4)$$

Each of the sets $\mathbb{C}(i_k)$ is isomorphic to the field of complex numbers, and \mathbb{D} is the set of *hyperbolic numbers*. An arbitrary bicomplex number w can be written as $w = z + z'i_2$, where $z = w_e + w_{i_1}i_1$ and $z' = w_{i_2} + w_ji_1$ both belong to $\mathbb{C}(i_1)$.

Bicomplex algebra is considerably simplified by the introduction of two bicomplex numbers \mathbf{e}_1 and \mathbf{e}_2 defined as

$$\mathbf{e}_1 := \frac{1+j}{2}, \quad \mathbf{e}_2 := \frac{1-j}{2}. \quad (2.5)$$

One easily checks that

$$\mathbf{e}_1^2 = \mathbf{e}_1, \quad \mathbf{e}_2^2 = \mathbf{e}_2, \quad \mathbf{e}_1 + \mathbf{e}_2 = 1, \quad \mathbf{e}_1 \mathbf{e}_2 = 0. \quad (2.6)$$

Any bicomplex number w can be written uniquely as

$$w = z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2, \quad (2.7)$$

where z_1 and z_2 both belong to $\mathbb{C}(i_1)$. Specifically,

$$z_1 = (w_e + w_j) + (w_{i_1} - w_{i_2})i_1, \quad z_2 = (w_e - w_j) + (w_{i_1} + w_{i_2})i_1. \quad (2.8)$$

The numbers \mathbf{e}_1 and \mathbf{e}_2 make up the so-called *idempotent basis* of the bicomplex numbers. Note that the last of (2.6) illustrates the fact that \mathbb{T} has zero divisors which, we recall, are non-zero elements whose product is zero.

The product of two bicomplex numbers w and w' can be written in the idempotent basis as

$$w \cdot w' = (z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2) \cdot (z'_1 \mathbf{e}_1 + z'_2 \mathbf{e}_2) = z_1 z'_1 \mathbf{e}_1 + z_2 z'_2 \mathbf{e}_2. \quad (2.9)$$

Since 1 is uniquely decomposed as $\mathbf{e}_1 + \mathbf{e}_2$, we can see that $w \cdot w' = 1$ if and only if $z_1 z'_1 = 1 = z_2 z'_2$. Thus w has an inverse if and only if $z_1 \neq 0 \neq z_2$, and the inverse w^{-1} is then equal to $z_1^{-1} \mathbf{e}_1 + z_2^{-1} \mathbf{e}_2$. A non-zero w that does not have an inverse has the property that either $z_1 = 0$ or $z_2 = 0$, and such a w is a divisor of zero. Zero divisors make up the so-called null cone \mathcal{NC} . That terminology comes from the fact that when w is written as $z + z' i_2$, zero divisors are such that $z^2 + (z')^2 = 0$.

With w written as in (2.7), we define two projection operators P_1 and P_2 so that

$$P_1(w) = z_1, \quad P_2(w) = z_2. \quad (2.10)$$

One can easily check that, for $k = 1, 2$,

$$[P_k]^2 = P_k, \quad \mathbf{e}_1 P_1 + \mathbf{e}_2 P_2 = \text{Id} \quad (2.11)$$

and that, for any $s, t \in \mathbb{T}$,

$$P_k(s + t) = P_k(s) + P_k(t), \quad P_k(s \cdot t) = P_k(s) \cdot P_k(t). \quad (2.12)$$

We define the conjugate w^\dagger of the bicomplex number $w = z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2$ as

$$w^\dagger := \bar{z}_1 \mathbf{e}_1 + \bar{z}_2 \mathbf{e}_2, \quad (2.13)$$

where the bar denotes the usual complex conjugation. Operation w^\dagger was denoted by $w^{\dagger 3}$ in [14, 17], consistent with the fact that at least two other types of conjugation can be defined with bicomplex numbers. Making use of (2.9) we immediately see that

$$w \cdot w^\dagger = z_1 \bar{z}_1 \mathbf{e}_1 + z_2 \bar{z}_2 \mathbf{e}_2. \quad (2.14)$$

Furthermore, for any $s, t \in \mathbb{T}$,

$$(s + t)^\dagger = s^\dagger + t^\dagger, \quad (s^\dagger)^\dagger = s, \quad (s \cdot t)^\dagger = s^\dagger \cdot t^\dagger. \quad (2.15)$$

The real modulus $|w|$ of a bicomplex number w can be defined as

$$|w| := \sqrt{w_e^2 + w_{i_1}^2 + w_{i_2}^2 + w_j^2} = \sqrt{(z_1 \bar{z}_1 + z_2 \bar{z}_2)/2}. \quad (2.16)$$

This coincides with the Euclidean norm on \mathbb{R}^4 . Clearly, $|w| \geq 0$, with $|w| = 0$ if and only if $w = 0$. Moreover, one can show [16] that for any $s, t \in \mathbb{T}$,

$$|s + t| \leq |s| + |t|, \quad |s \cdot t| \leq \sqrt{2}|s| \cdot |t|. \quad (2.17)$$

The modulus $|w|$ satisfies the first four properties of the absolute value N introduced in section 1, but it fails to satisfy the fifth one. The modulus can be redefined so as to eliminate the $\sqrt{2}$ in (2.17), but we are following Price's conventions [15].

In the idempotent basis, any hyperbolic number can be written as $x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$, with x_1 and x_2 in \mathbb{R} . We define the set \mathbb{D}^+ of positive hyperbolic numbers as

$$\mathbb{D}^+ := \{x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 \mid x_1, x_2 \in \mathbb{R}^+\}. \quad (2.18)$$

Clearly, $w \cdot w^\dagger \in \mathbb{D}^+$ for any w in \mathbb{T} . We shall say that w is in $\mathbf{e}_1 \mathbb{R}^+$ if $w = x_1 \mathbf{e}_1$ and x_1 is in \mathbb{R}^+ (and similarly with $\mathbf{e}_2 \mathbb{R}^+$).

2.2 Modules, scalar product and linear operators

By definition, a vector space is specified over a field of numbers. Bicomplex numbers make up a ring rather than a field, and the structure analogous to a vector space is then a *module*. For later reference we define a \mathbb{T} -module M as a set of elements $|\psi\rangle, |\phi\rangle, |\chi\rangle, \dots$, endowed with operations of addition and scalar multiplication, such that the following always holds:

- i. $|\psi\rangle + |\phi\rangle = |\phi\rangle + |\psi\rangle$;
- ii. $(|\psi\rangle + |\phi\rangle) + |\chi\rangle = |\psi\rangle + (|\phi\rangle + |\chi\rangle)$;
- iii. there exists a $|0\rangle$ in M such that $|0\rangle + |\psi\rangle = |\psi\rangle$;
- iv. $0 \cdot |\psi\rangle = |0\rangle$;
- v. $1 \cdot |\psi\rangle = |\psi\rangle$;
- vi. $s \cdot (|\psi\rangle + |\phi\rangle) = s \cdot |\psi\rangle + s \cdot |\phi\rangle$;

$$\text{vii. } (s + t) \cdot |\psi\rangle = s \cdot |\psi\rangle + t \cdot |\psi\rangle;$$

$$\text{viii. } (st) \cdot |\psi\rangle = s \cdot (t \cdot |\psi\rangle).$$

Here $s, t \in \mathbb{T}$. We have introduced Dirac's notation for elements of M , which we shall call *kets* even though they are not genuine vectors.

A finite-dimensional *free* \mathbb{T} -module is a \mathbb{T} -module with a finite basis, *i.e.* a finite set of linearly independent elements that generate the module. Explicitly, M is a finite-dimensional free \mathbb{T} -module if there exist n linearly independent kets $|u_l\rangle$ such that any element $|\psi\rangle$ of M can be written as

$$|\psi\rangle = \sum_{l=1}^n w_l |u_l\rangle, \quad (2.19)$$

with $w_l \in \mathbb{T}$. An important subset V of M is the set of all kets for which all w_l in (2.19) belong to $\mathbb{C}(i_1)$. It was shown in [14] that V is a vector space over the complex numbers, and that any $|\psi\rangle \in M$ can be decomposed uniquely as

$$|\psi\rangle = \mathbf{e}_1 P_1(|\psi\rangle) + \mathbf{e}_2 P_2(|\psi\rangle), \quad (2.20)$$

where P_1 and P_2 are projectors from M to V . Note that V depends on the basis $\{|u_l\rangle\}$. One can show that ket projectors and idempotent-basis projectors (denoted with the same symbol) satisfy the following, for $k = 1, 2$:

$$P_k(s|\psi\rangle + t|\phi\rangle) = P_k(s)P_k(|\psi\rangle) + P_k(t)P_k(|\phi\rangle). \quad (2.21)$$

It will be very useful to rewrite (2.7) and (2.20) as

$$w = w_1 + w_2, \quad |\psi\rangle = |\psi\rangle_1 + |\psi\rangle_2, \quad (2.22)$$

where

$$w_1 = \mathbf{e}_1 z_1, \quad w_2 = \mathbf{e}_2 z_2, \quad |\psi\rangle_1 = \mathbf{e}_1 P_1(|\psi\rangle), \quad |\psi\rangle_2 = \mathbf{e}_2 P_2(|\psi\rangle). \quad (2.23)$$

Henceforth bold indices (like **1** and **2**) will always denote objects which include a factor \mathbf{e}_1 or \mathbf{e}_2 , and therefore satisfy an equation like $w_1 = \mathbf{e}_1 w_1$.

A *bicomplex scalar product* maps two arbitrary kets $|\psi\rangle$ and $|\phi\rangle$ into a bicomplex number $(|\psi\rangle, |\phi\rangle)$, so that the following always holds ($s \in \mathbb{T}$):

- i. $(|\psi\rangle, |\phi\rangle + |\chi\rangle) = (|\psi\rangle, |\phi\rangle) + (|\psi\rangle, |\chi\rangle);$
- ii. $(|\psi\rangle, s|\phi\rangle) = s(|\psi\rangle, |\phi\rangle);$
- iii. $(|\psi\rangle, |\phi\rangle) = (|\phi\rangle, |\psi\rangle)^\dagger;$

$$\text{iv. } (|\psi\rangle, |\psi\rangle) = 0 \Leftrightarrow |\psi\rangle = 0.$$

Property (iii) implies that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}$. In fact we will use in sections 3 and 4 the stronger requirement that $(|\psi\rangle, |\psi\rangle) \in \mathbb{D}^+$. Writing property (iii) with one of the other types of conjugation mentioned after (2.13) would make $(|\psi\rangle, |\psi\rangle)$ belong to $\mathbb{C}(i_1)$ or $\mathbb{C}(i_2)$.

Properties (ii) and (iii) together imply that $(s|\psi\rangle, |\phi\rangle) = s^\dagger(|\psi\rangle, |\phi\rangle)$. One easily shows that

$$(|\psi\rangle, |\phi\rangle) = (|\psi\rangle_1, |\phi\rangle_1) + (|\psi\rangle_2, |\phi\rangle_2). \quad (2.24)$$

Note that

$$(|\psi\rangle_1, |\phi\rangle_1)_1 = (|\psi\rangle_1, |\phi\rangle_1) \quad \text{and} \quad (|\psi\rangle_2, |\phi\rangle_2)_2 = (|\psi\rangle_2, |\phi\rangle_2). \quad (2.25)$$

A *bicomplex linear operator* A is a mapping from M to M such that, for any $s, t \in \mathbb{T}$ and any $|\psi\rangle, |\phi\rangle \in M$

$$A(s|\psi\rangle + t|\phi\rangle) = sA|\psi\rangle + tA|\phi\rangle. \quad (2.26)$$

The bicomplex *adjoint* operator A^* of A is the operator defined so that for any $|\psi\rangle, |\phi\rangle \in M$

$$(|\psi\rangle, A|\phi\rangle) = (A^*|\psi\rangle, |\phi\rangle). \quad (2.27)$$

One can show that in finite-dimensional free \mathbb{T} -modules, the adjoint always exists, is linear and satisfies

$$(A^*)^* = A, \quad (sA + tB)^* = s^\dagger A^* + t^\dagger B^*, \quad (AB)^* = B^* A^*. \quad (2.28)$$

A bicomplex linear operator A can always be written as $A = A_1 + A_2$, with $A_1 = \mathbf{e}_1 A$ and $A_2 = \mathbf{e}_2 A$. Clearly,

$$A|\psi\rangle = A_1|\psi\rangle_1 + A_2|\psi\rangle_2. \quad (2.29)$$

We shall say that a ket $|\psi\rangle$ belongs to the null cone if either $|\psi\rangle_1 = 0$ or $|\psi\rangle_2 = 0$, and that a linear operator A belongs to the null cone if either $A_1 = 0$ or $A_2 = 0$.

A *self-adjoint* operator is a linear operator H such that $H = H^*$. From (2.27) one sees immediately that H is self-adjoint if and only if

$$(|\psi\rangle, H|\phi\rangle) = (H|\psi\rangle, |\phi\rangle) \quad (2.30)$$

for all $|\psi\rangle$ and $|\phi\rangle$ in M .

It was shown in [14] that the eigenvalues of a self-adjoint operator acting in a finite-dimensional free \mathbb{T} -module, associated with eigenkets not in the null cone, are hyperbolic numbers. One can show quite straightforwardly that two such eigenkets of such a self-adjoint operator, whose eigenvalues differ by a quantity that is not in the null cone, are orthogonal. The proof of this statement can be found as part of a detailed study of finite-dimensional free \mathbb{T} -modules [18].

3 The harmonic oscillator

The harmonic oscillator is one of the most widely discussed and widely applied problems in standard quantum mechanics. It is specified as follows: Find the eigenvalues and eigenvectors of a self-adjoint operator H defined as

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2, \quad (3.1)$$

where m and ω are positive real numbers and X and P are self-adjoint operators satisfying the following commutation relation (with i_1 the usual imaginary i):

$$[X, P] = i_1 \hbar I. \quad (3.2)$$

The problem can be solved exactly either by algebraic [19, 20] or differential [21] methods. In this section we shall show that, viewed as an algebraic problem, the standard quantum-mechanical harmonic oscillator generalizes to bicomplex numbers. One of the advantages of the algebraic method is that the uniqueness of the structure determined by the assumptions is quite transparent. In the process we shall build explicitly an example of an infinite-dimensional free \mathbb{T} -module.

3.1 Definitions and assumptions

To state and solve the problem of the bicomplex quantum harmonic oscillator, we start with the following assumptions:

- a. Three linear operators X , P and H , related by (3.1), act in a free \mathbb{T} -module M .
- b. X , P and H are self-adjoint with respect to a scalar product yet to be defined. This means that $(|\psi\rangle, H|\phi\rangle) = (H|\psi\rangle, |\phi\rangle)$ for any $|\psi\rangle$ and $|\phi\rangle$ in M on which H is defined, and similarly with X and P .
- c. The scalar product of a ket with itself belongs to \mathbb{D}^+ .
- d. $[X, P] = i_1 \hbar \xi I$, where $\xi \in \mathbb{T}$ is not in the null cone and I is the identity operator on M .
- e. There is at least one normalizable eigenket $|E\rangle$ of H which is not in the null cone and whose corresponding eigenvalue E is not in the null cone.
- f. Eigenkets of H that are not in the null cone and that correspond to eigenvalues whose difference is not in the null cone are orthogonal.

The consistency of these assumptions will be verified explicitly once the full structure has been obtained. Note that not all of them are expected to hold in an arbitrary infinite-dimensional module, but they do in the one we are going to construct. The simplest extension of the canonical commutation relations seems to be embodied in (d). Note that (d) implies that neither X nor P are in the null cone, for if one of them were, ξ would also belong to \mathcal{NC} . Assumption (e) implies that H is not in the null cone, and this is necessary to end up with a non-trivial generalization of the standard quantum-mechanical case.

The self-adjointness of X and P implies that the bicomplex number ξ in (d) is in fact hyperbolic. Indeed let $|E\rangle$ be the eigenket of H introduced in (e). By the properties of the scalar product and definition of self-adjointness,

$$\begin{aligned} \mathbf{i}_1 \hbar \xi (|E\rangle, |E\rangle) &= (|E\rangle, \mathbf{i}_1 \hbar \xi I |E\rangle) = (|E\rangle, XP|E\rangle) - (|E\rangle, PX|E\rangle) \\ &= (X|E\rangle, P|E\rangle) - (P|E\rangle, X|E\rangle) \\ &= (PX|E\rangle, |E\rangle) - (XP|E\rangle, |E\rangle) \\ &= (-\mathbf{i}_1 \hbar \xi I |E\rangle, |E\rangle) = \mathbf{i}_1 \hbar \xi^\dagger (|E\rangle, |E\rangle). \end{aligned} \quad (3.3)$$

Since $|E\rangle$ is normalizable, $(|E\rangle, |E\rangle)$ is not in the null cone, and it immediately follows that $\xi = \xi^\dagger$. That is, $\xi = \xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2$, with ξ_1 and ξ_2 real.

Is it possible to further restrict meaningful values of ξ , for instance by a simple rescaling of X and P ? To answer this question, let us write

$$X = (\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2) X', \quad P = (\beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2) P', \quad (3.4)$$

with non-zero α_k and β_k ($k = 1, 2$). For X' and P' to be self-adjoint, α_k and β_k must be real. Making use of (3.1) we find that

$$\begin{aligned} H &= \frac{1}{2m} (\beta_1^2 \mathbf{e}_1 + \beta_2^2 \mathbf{e}_2) (P')^2 + \frac{1}{2} m \omega^2 (\alpha_1^2 \mathbf{e}_1 + \alpha_2^2 \mathbf{e}_2) (X')^2 \\ &= \frac{1}{2m'} (P')^2 + \frac{1}{2} m' (\omega')^2 (X')^2. \end{aligned} \quad (3.5)$$

For m' and ω' to be positive real numbers, $\alpha_1^2 \mathbf{e}_1 + \alpha_2^2 \mathbf{e}_2$ and $\beta_1^2 \mathbf{e}_1 + \beta_2^2 \mathbf{e}_2$ must also belong to \mathbb{R}^+ . This entails that $\alpha_1^2 = \alpha_2^2$ and $\beta_1^2 = \beta_2^2$, or equivalently $\alpha_1 = \pm \alpha_2$ and $\beta_1 = \pm \beta_2$. Hence we can write

$$\begin{aligned} \mathbf{i}_1 \hbar (\xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2) I &= [X, P] \\ &= [(\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2) X', (\beta_1 \mathbf{e}_1 + \beta_2 \mathbf{e}_2) P'] \\ &= (\alpha_1 \beta_1 \mathbf{e}_1 + \alpha_2 \beta_2 \mathbf{e}_2) [X', P']. \end{aligned} \quad (3.6)$$

But this in turn implies that

$$[X', P'] = \mathbf{i}_1 \hbar \left(\frac{\xi_1}{\alpha_1 \beta_1} \mathbf{e}_1 + \frac{\xi_2}{\alpha_2 \beta_2} \mathbf{e}_2 \right) I = \mathbf{i}_1 \hbar (\xi'_1 \mathbf{e}_1 + \xi'_2 \mathbf{e}_2) I. \quad (3.7)$$

This equation shows that α_1 , α_2 , β_1 and β_2 can always be picked so that ξ'_1 and ξ'_2 are positive. Furthermore, we can choose α_1 and β_1 so as to make ξ'_1 equal to 1. But since $|\alpha_1\beta_1| = |\alpha_2\beta_2|$, we have no control over the norm of ξ'_2 . The upshot is that we can always write H as in (3.1), with the commutation relation of X and P given by

$$[X, P] = i_1 \hbar \xi I = i_1 \hbar (\xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2) I, \quad \xi_1, \xi_2 \in \mathbb{R}^+. \quad (3.8)$$

We also have the freedom of setting either $\xi_1 = 1$ or $\xi_2 = 1$, but not both.

Just as in the case of the standard quantum harmonic oscillator, we now introduce two operators A and A^* as

$$A := \frac{1}{\sqrt{2m\hbar\omega}}(m\omega X + i_1 P), \quad (3.9)$$

$$A^* := \frac{1}{\sqrt{2m\hbar\omega}}(m\omega X - i_1 P). \quad (3.10)$$

Since P is self-adjoint, one always has $(-i_1 P|\psi\rangle, |\phi\rangle) = (|\psi\rangle, i_1 P|\phi\rangle)$, which means that the adjoint of $i_1 P$ is $-i_1 P$. This implies that, as the notation suggests, A^* is indeed the adjoint of A . Equations (3.9) and (3.10) can be inverted as

$$X = \sqrt{\frac{\hbar}{2m\omega}}(A + A^*), \quad P = -i_1 \sqrt{\frac{\hbar m\omega}{2}}(A - A^*). \quad (3.11)$$

The commutator of A and A^* is given by

$$[A, A^*] = \frac{1}{2m\hbar\omega} \{[i_1 P, m\omega X] + [m\omega X, -i_1 P]\} = \xi I. \quad (3.12)$$

Substituting (3.11) in (3.1), one easily finds that

$$H = \hbar\omega \left(A^* A + \frac{\xi}{2} I \right) = \hbar\omega \left(A A^* - \frac{\xi}{2} I \right). \quad (3.13)$$

From (3.12) and (3.13), the following commutation relations are straightforwardly obtained:

$$[H, A] = -\hbar\omega \xi A, \quad [H, A^*] = \hbar\omega \xi A^*. \quad (3.14)$$

Our problem is therefore the following: Find the eigenvalues and eigenkets of the Hamiltonian (3.13), subject to the constraint (3.12) and, consequently, (3.14).

3.2 Eigenkets and eigenvalues of H

From assumption (e) we know that there is a normalizable ket $|E\rangle$ such that

$$H|E\rangle = E|E\rangle. \quad (3.15)$$

We can write

$$H = H_1 + H_2, \quad (3.16)$$

$$E = E_1 + E_2, \quad (3.17)$$

$$|E\rangle = |E\rangle_1 + |E\rangle_2, \quad (3.18)$$

where $E_1 = \mathbf{e}_1 E$, etc. Assumption (e) implies that none of the quantities in (3.16)–(3.18) vanishes. Substitution of these equations in (3.15) immediately yields

$$H_1|E\rangle_1 = E_1|E\rangle_1, \quad H_2|E\rangle_2 = E_2|E\rangle_2. \quad (3.19)$$

Following the treatment made in standard quantum mechanics, we now apply operators HA and HA^* on $|E\rangle$. Making use of (3.14) we get

$$HA|E\rangle = (AH + [H, A])|E\rangle = (E - \hbar\omega\xi)A|E\rangle, \quad (3.20)$$

$$HA^*|E\rangle = (A^*H + [H, A^*])|E\rangle = (E + \hbar\omega\xi)A^*|E\rangle. \quad (3.21)$$

We see that if $A|E\rangle$ does not vanish, it is an eigenket of H with eigenvalue $E - \hbar\omega\xi$. Similarly, unless $A^*|E\rangle$ vanishes, it is an eigenket of H with eigenvalue $E + \hbar\omega\xi$.

Let l be a positive integer. We will show by induction that unless $A^l|E\rangle$ vanishes, it is an eigenket of H with eigenvalue $E - l\hbar\omega\xi$. We have just shown that this is true for $l = 1$. Let it be true for $l - 1$. We have

$$\begin{aligned} HA^l|E\rangle &= HAA^{l-1}|E\rangle = (AHA^{l-1} + [H, A]A^{l-1})|E\rangle \\ &= A(E - (l-1)\hbar\omega\xi)A^{l-1}|E\rangle - \hbar\omega\xi AA^{l-1}|E\rangle \\ &= \{E - l\hbar\omega\xi\} A^l|E\rangle, \end{aligned} \quad (3.22)$$

which proves the claim. Similarly, unless $(A^*)^l|E\rangle$ vanishes, it is an eigenket of H with eigenvalue $E + l\hbar\omega\xi$, that is,

$$H(A^*)^l|E\rangle = (E + l\hbar\omega\xi)(A^*)^l|E\rangle. \quad (3.23)$$

Equations (3.22) and (3.23) separate in the idempotent basis. Multiplying them by \mathbf{e}_k and using the fact that $HA^l = H_1A_1^l + H_2A_2^l$, we easily find that ($k = 1, 2$)

$$H_k A_k^l |E\rangle_k = (E_k - l\hbar\omega\xi_k) A_k^l |E\rangle_k, \quad (3.24)$$

$$H_k (A_k^*)^l |E\rangle_k = (E_k + l\hbar\omega\xi_k) (A_k^*)^l |E\rangle_k. \quad (3.25)$$

Consistent with the bold notation, we have written $\xi_1 = \mathbf{e}_1 \xi_1$ and $\xi_2 = \mathbf{e}_2 \xi_2$.

We now prove the following lemma.

Lemma 1 *Let $|\phi\rangle$ be an eigenket of H associated with the (finite) eigenvalue λ . Then,*

$$(A|\phi\rangle, A|\phi\rangle) = \left\{ \frac{\lambda}{\hbar\omega} - \frac{\xi}{2} \right\} (|\phi\rangle, |\phi\rangle) \quad (3.26)$$

and

$$(A^*|\phi\rangle, A^*|\phi\rangle) = \left\{ \frac{\lambda}{\hbar\omega} + \frac{\xi}{2} \right\} (|\phi\rangle, |\phi\rangle). \quad (3.27)$$

Proof.

Making use of (3.13) we have

$$\begin{aligned} (A|\phi\rangle, A|\phi\rangle) &= (|\phi\rangle, A^*A|\phi\rangle) = \left(|\phi\rangle, \left\{ \frac{H}{\hbar\omega} - \frac{\xi}{2}I \right\} |\phi\rangle \right) \\ &= \left(|\phi\rangle, \left\{ \frac{\lambda}{\hbar\omega} - \frac{\xi}{2} \right\} |\phi\rangle \right) = \left\{ \frac{\lambda}{\hbar\omega} - \frac{\xi}{2} \right\} (|\phi\rangle, |\phi\rangle). \end{aligned}$$

The proof of the second equality is similar. \square

Two important consequences of lemma 1 are the following. Firstly, whenever $(|\phi\rangle, |\phi\rangle)$ is finite, so are $(A|\phi\rangle, A|\phi\rangle)$ and $(A^*|\phi\rangle, A^*|\phi\rangle)$. And secondly, the lemma also holds when all quantities are replaced by corresponding idempotent projections. That is, for $k = 1, 2$,

$$(A_{\mathbf{k}}|\phi\rangle_{\mathbf{k}}, A_{\mathbf{k}}|\phi\rangle_{\mathbf{k}}) = \left\{ \frac{\lambda_{\mathbf{k}}}{\hbar\omega} - \frac{\xi_{\mathbf{k}}}{2} \right\} (|\phi\rangle_{\mathbf{k}}, |\phi\rangle_{\mathbf{k}}). \quad (3.28)$$

Let us now apply lemma 1 to the case where $|\phi\rangle_{\mathbf{k}} = |E\rangle_{\mathbf{k}}$. Since $(|E\rangle, |E\rangle)$ is in \mathbb{D}^+ , $(|E\rangle_{\mathbf{k}}, |E\rangle_{\mathbf{k}})$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$ (and is non-zero). But then (3.28) implies that $(A_{\mathbf{k}}|E\rangle_{\mathbf{k}}, A_{\mathbf{k}}|E\rangle_{\mathbf{k}})$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$ only if $E_{\mathbf{k}}/\hbar\omega - \xi_{\mathbf{k}}/2$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$. Let us write (3.28) for the case where $|\phi\rangle_{\mathbf{k}} = A_{\mathbf{k}}^l|E\rangle_{\mathbf{k}}$. Making use of (3.24), we find that

$$(A_{\mathbf{k}}^{l+1}|E\rangle_{\mathbf{k}}, A_{\mathbf{k}}^{l+1}|E\rangle_{\mathbf{k}}) = \left\{ \frac{E_{\mathbf{k}}}{\hbar\omega} - \left(l + \frac{1}{2} \right) \xi_{\mathbf{k}} \right\} (A_{\mathbf{k}}^l|E\rangle_{\mathbf{k}}, A_{\mathbf{k}}^l|E\rangle_{\mathbf{k}}). \quad (3.29)$$

Again, and assuming that $A_{\mathbf{k}}^l|E\rangle_{\mathbf{k}}$ doesn't vanish, $(A_{\mathbf{k}}^{l+1}|E\rangle_{\mathbf{k}}, A_{\mathbf{k}}^{l+1}|E\rangle_{\mathbf{k}})$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$ only if $E_{\mathbf{k}}/\hbar\omega - (l + 1/2)\xi_{\mathbf{k}}$ is in $\mathbf{e}_{\mathbf{k}}\mathbb{R}^+$.

Clearly, however, this cannot go on forever. Let l_k be the smallest positive integer for which

$$P_k \left(\frac{E_{\mathbf{k}}}{\hbar\omega} - \left(l_k + \frac{1}{2} \right) \xi_{\mathbf{k}} \right) \leq 0. \quad (3.30)$$

If the equality holds in (3.30), then (3.29) implies that $A_{\mathbf{k}}^{l_k+1}|E\rangle_{\mathbf{k}} = 0$. If the inequality holds, the same conclusion follows since otherwise the scalar product of a non-zero ket with itself would be outside \mathbb{D}^+ . The upshot is that

$$A_{\mathbf{k}}|\phi_0\rangle_{\mathbf{k}} = 0 \quad \text{with} \quad |\phi_0\rangle_{\mathbf{k}} = A_{\mathbf{k}}^{l_k}|E\rangle_{\mathbf{k}}. \quad (3.31)$$

Applying $H_{\mathbf{k}}$ obtained from the first part of (3.13) on $|\phi_0\rangle_{\mathbf{k}}$, we get

$$H_{\mathbf{k}}|\phi_0\rangle_{\mathbf{k}} = \hbar\omega \left(A_{\mathbf{k}}^* A_{\mathbf{k}} + \frac{1}{2} \xi_{\mathbf{k}} I \right) |\phi_0\rangle_{\mathbf{k}} = \frac{1}{2} \hbar\omega \xi_{\mathbf{k}} |\phi_0\rangle_{\mathbf{k}}. \quad (3.32)$$

That is, $|\phi_0\rangle_{\mathbf{k}}$ is an eigenket of $H_{\mathbf{k}}$ with eigenvalue $\hbar\omega \xi_{\mathbf{k}}/2$.

Making use of an argument similar to the one leading to (3.25), we can see that the ket $(A_{\mathbf{k}}^*)^l |\phi_0\rangle_{\mathbf{k}}$ is an eigenket of $H_{\mathbf{k}}$ with eigenvalue $(l + 1/2)\hbar\omega \xi_{\mathbf{k}}$. For later convenience we define

$$|\phi_l\rangle_{\mathbf{1}} = (l! \xi_1^l)^{-1/2} (A_{\mathbf{1}}^*)^l |\phi_0\rangle_{\mathbf{1}}, \quad |\phi_l\rangle_{\mathbf{2}} = (l! \xi_2^l)^{-1/2} (A_{\mathbf{2}}^*)^l |\phi_0\rangle_{\mathbf{2}}. \quad (3.33)$$

Note that ξ_1 and ξ_2 , being within an inversion operator, cannot carry bold indices. By the idempotent projection of the second part of lemma 1, $|\phi_l\rangle_{\mathbf{k}}$ does not vanish for any l . We have therefore constructed two infinite sequences of kets, each of which is a sequence of eigenkets of an idempotent projection of H .

We now define

$$|\phi_l\rangle = |\phi_l\rangle_{\mathbf{1}} + |\phi_l\rangle_{\mathbf{2}}. \quad (3.34)$$

It is easy to check that $|\phi_l\rangle$ is an eigenket of H with eigenvalue $(l + 1/2)\hbar\omega \xi$. By assumption (f), $|\phi_l\rangle$ and $|\phi_{l'}\rangle$ are orthogonal if $l \neq l'$.

3.3 Infinite-dimensional free \mathbb{T} -module

Let M be the collection of all finite linear combinations of kets $|\phi_l\rangle$, with bicomplex coefficients. That is,

$$M := \left\{ \sum_l w_l |\phi_l\rangle \mid w_l \in \mathbb{T} \right\}. \quad (3.35)$$

It is understood that adding terms with zero coefficients doesn't yield a new ket. Let us define the addition of two elements of M and the multiplication of an element of M by a bicomplex number in the obvious way. Furthermore let us write $|0\rangle = 0 \cdot |\phi_0\rangle$. It is then easy to check that the eight defining properties of a \mathbb{T} -module stated in section 2.2 are satisfied. M is therefore a \mathbb{T} -module. Clearly, H is defined everywhere on M .

If the coefficients w_l in (3.35) are restricted to elements of $\mathbb{C}(\mathbf{i}_1)$, the resulting set V is a vector space over $\mathbb{C}(\mathbf{i}_1)$. It is the analog of the vector space introduced before (2.20), which was used in [14] to define the projection P_k and prove a number of results on finite-dimensional modules.

The scalar product of elements of M has hitherto been specified only partially, in particular by requiring that $|\phi_l\rangle$ and $|\phi_{l'}\rangle$ be orthogonal if $l \neq l'$. We now set

$$(|\phi_0\rangle, |\phi_0\rangle) = 1. \quad (3.36)$$

Equation (3.33) implies that

$$\begin{aligned}
|\phi_{l+1}\rangle &= |\phi_{l+1}\rangle_1 + |\phi_{l+1}\rangle_2 = \mathbf{e}_1 |\phi_{l+1}\rangle_1 + \mathbf{e}_2 |\phi_{l+1}\rangle_2 \\
&= \frac{\mathbf{e}_1}{\sqrt{(l+1)\xi_1}} A_1^* |\phi_l\rangle_1 + \frac{\mathbf{e}_2}{\sqrt{(l+1)\xi_2}} A_2^* |\phi_l\rangle_2 \\
&= \frac{1}{\sqrt{(l+1)\xi}} A^* |\phi_l\rangle.
\end{aligned} \tag{3.37}$$

Letting A act on both sides of (3.37) and making use of (3.13), we find that

$$\begin{aligned}
A|\phi_{l+1}\rangle &= \frac{1}{\sqrt{(l+1)\xi}} AA^* |\phi_l\rangle = \frac{1}{\sqrt{(l+1)\xi}} \left\{ \frac{H}{\hbar\omega} + \frac{\xi}{2} I \right\} |\phi_l\rangle \\
&= \sqrt{(l+1)\xi} |\phi_l\rangle.
\end{aligned} \tag{3.38}$$

Equations (3.37) and (3.38) imply that A and A^* , and therefore X and P , are defined everywhere on M . This means that all quantities involved, for instance, in (3.3) are well-defined.

From (3.37) and the second part of lemma 1 we get

$$(|\phi_{l+1}\rangle, |\phi_{l+1}\rangle) = \frac{1}{(l+1)\xi} (A^* |\phi_l\rangle, A^* |\phi_l\rangle) = (|\phi_l\rangle, |\phi_l\rangle). \tag{3.39}$$

Owing to (3.36), the solution of this recurrence equation is

$$(|\phi_l\rangle, |\phi_l\rangle) = 1, \quad l = 0, 1, 2, \dots \tag{3.40}$$

We now fully specify the scalar product of two arbitrary elements $|\psi\rangle$ and $|\chi\rangle$ of M as follows. Let

$$|\psi\rangle = \sum_l w_l |\phi_l\rangle, \quad |\chi\rangle = \sum_l v_l |\phi_l\rangle. \tag{3.41}$$

The two sums are finite. Without loss of generality, we can let them run over the same set of indices. Indeed this simply amounts to possibly adding terms with zero coefficients in either or both sums. With this we define the scalar product as

$$(|\psi\rangle, |\chi\rangle) := \sum_l w_l^\dagger v_l (|\phi_l\rangle, |\phi_l\rangle) = \sum_l w_l^\dagger v_l. \tag{3.42}$$

With this specification, it is easy to check that the four defining properties of a scalar product stated in section 2.2 are satisfied. Note that the right-hand side of (3.42) is always finite.

Clearly, the kets $|\phi_l\rangle$ generate M . To show that they are linearly independent, we assume that $|\psi\rangle$ defined in (3.41) vanishes. Letting m be one of the l indices, we have

$$0 = (|\phi_m\rangle, |\psi\rangle) = \sum_l w_l \delta_{ml} = w_m. \quad (3.43)$$

Hence $w_m = 0$ for all m , and the linear independence follows. This shows that M is an infinite-dimensional free \mathbb{T} -module.

There remains to check the six assumptions made at the beginning of section 3.1. Assumption (a) is obvious, the action of X and P on M being most easily obtained through the action of A and A^* . Similarly with (b), the self-adjointness of X and P follows from the easily verifiable fact that A^* is the adjoint of A on the whole of M . Assumption (c) is an immediate consequence of definition (3.42). Assumption (d) follows from the commutation relation $[A, A^*] = \xi I$. This one is easily checked when acting on eigenkets of H and therefore, by linearity, it holds on any ket. Assumption (e) is satisfied by any ket $|\phi_l\rangle$. There only remains to check assumption (f), which is a little more tricky.

Let $|\psi\rangle$ defined in (3.41) be an eigenket of H with eigenvalue λ . This means that

$$H \sum_l w_l |\phi_l\rangle = \lambda \sum_l w_l |\phi_l\rangle \quad (3.44)$$

which, owing to the linear independence of the $|\phi_l\rangle$, reduces to

$$\left(l + \frac{1}{2}\right) \hbar \omega \xi w_l = \lambda w_l. \quad (3.45)$$

In the idempotent basis this becomes ($k = 1, 2$)

$$\left(l + \frac{1}{2}\right) \hbar \omega \xi_{\mathbf{k}} w_{l\mathbf{k}} = \lambda_{\mathbf{k}} w_{l\mathbf{k}}. \quad (3.46)$$

Let $\lambda_{\mathbf{1}} \neq 0$. Since $\xi_{\mathbf{1}}$ does not vanish, at most one coefficient $w_{l\mathbf{1}}$ does not vanish, for otherwise $\lambda_{\mathbf{1}}$ would satisfy two incompatible equations. If $\lambda_{\mathbf{1}} = 0$, all $w_{l\mathbf{1}}$ vanish. A similar argument holds for $\mathbf{2}$. Hence the eigenket of H has the form

$$|\phi\rangle = w_{l\mathbf{1}} |\phi_l\rangle_{\mathbf{1}} + w_{l'\mathbf{2}} |\phi_{l'}\rangle_{\mathbf{2}}, \quad (3.47)$$

with one of the coefficients vanishing if the corresponding $\lambda_{\mathbf{k}}$ vanishes. If both $\lambda_{\mathbf{k}}$ vanish, all $w_{l\mathbf{k}} = 0$ and there is no eigenket. The upshot is that (3.47) represents the most general eigenket of H . Its associated eigenvalue λ is

$$\lambda = \hbar \omega \left\{ \left(l + \frac{1}{2}\right) \xi_{\mathbf{1}} \mathbf{e}_{\mathbf{1}} + \left(l' + \frac{1}{2}\right) \xi_{\mathbf{2}} \mathbf{e}_{\mathbf{2}} \right\}. \quad (3.48)$$

It is now a simple matter to check that assumption (f) is satisfied. Note that the restriction on the difference of eigenvalues cannot be dispensed with. Indeed the two kets

$$|\phi\rangle = |\phi_1\rangle_1 + |\phi_2\rangle_2, \quad |\phi'\rangle = |\phi_1\rangle_1 + |\phi_3\rangle_2 \quad (3.49)$$

are examples of eigenkets that correspond to different eigenvalues whose difference is in the null cone. Clearly, they are not orthogonal.

4 Harmonic oscillator wave functions

4.1 Bicomplex function space

Let n be a non-negative integer and let α be a positive real number. Consider the following function of a real variable x :

$$f_{n,\alpha}(x) := x^n \exp(-\alpha x^2). \quad (4.1)$$

Let S be the set of all finite linear combinations of functions $f_{n,\alpha}(x)$, with complex coefficients. Furthermore, let a bicomplex function $u(x)$ be defined as

$$u(x) = \mathbf{e}_1 u_1(x) + \mathbf{e}_2 u_2(x), \quad (4.2)$$

where u_1 and u_2 are both in S . It is then easy to check that the set of all functions $u(x)$ is a \mathbb{T} -module, which we shall denote by M_S .

Let $u(x)$ and $v(x)$ both belong to M_S . We define a mapping (u, v) of this pair of functions into \mathbb{D}^+ as follows:

$$(u, v) := \int_{-\infty}^{\infty} u^\dagger(x) v(x) dx = \int_{-\infty}^{\infty} [\mathbf{e}_1 \bar{u}_1(x) v_1(x) + \mathbf{e}_2 \bar{u}_2(x) v_2(x)] dx. \quad (4.3)$$

It is not hard to see that (4.3) is always finite and satisfies all the properties of a bicomplex scalar product.

Let $\xi \in \mathbb{D}^+$. We define two operators X and P that act on elements of M_S as follows:

$$X\{u(x)\} := xu(x), \quad P\{u(x)\} := -\mathbf{i}_1 \hbar \xi \frac{du(x)}{dx}. \quad (4.4)$$

It is not difficult to show that $[X, P] = \mathbf{i}_1 \hbar \xi I$. Note that

$$X\{f_{n,\alpha}(x)\} = f_{n+1,\alpha}(x), \quad (4.5)$$

$$P\{f_{n,\alpha}(x)\} = -\mathbf{i}_1 \hbar \xi \{n f_{n-1,\alpha}(x) - 2\alpha f_{n+1,\alpha}(x)\}. \quad (4.6)$$

From this we conclude that the action of X and P on elements of M_S always yields elements of M_S (the function $f_{-1,\alpha}$, if any, coming with a vanishing coefficient). That is, X and P are defined on all M_S .

One can easily check that $(Xu, v) = (u, Xv)$, so that X is self-adjoint. The self-adjointness of P can be proved as

$$\begin{aligned}
(Pu, v) - (u, Pv) &= \int_{-\infty}^{\infty} \left(-i_1 \hbar \xi \frac{du(x)}{dx} \right)^\dagger v(x) dx - \int_{-\infty}^{\infty} u^\dagger(x) \left(-i_1 \hbar \xi \frac{dv(x)}{dx} \right) dx \\
&= i_1 \hbar \xi \left\{ \int_{-\infty}^{\infty} \frac{d[u^\dagger(x)v(x)]}{dx} dx \right\} \\
&= i_1 \hbar \xi [u^\dagger(x)v(x)]_{-\infty}^{\infty} = 0.
\end{aligned}$$

The final equality comes from the fact that u and v , involving finite sums of functions $f_{n,\alpha}(x)$, vanish at infinity.

4.2 Eigenfunctions of H

Let H be defined as in (3.1), with X and P specified as in (4.4). The eigenvalue equation for H is then given by

$$Hu(x) = -\frac{\hbar^2 \xi^2}{2m} \frac{d^2 u(x)}{dx^2} + \frac{1}{2} m \omega^2 x^2 u(x) = Eu(x). \quad (4.7)$$

In the idempotent basis this separates into the following two equations ($k = 1, 2$):

$$-\frac{\hbar^2 \xi_k^2}{2m} \frac{d^2 u_k(x)}{dx^2} + \frac{1}{2} m \omega^2 x^2 u_k(x) = E_k u_k(x). \quad (4.8)$$

Each of these equations is essentially the eigenvalue equation for the Hamiltonian of the standard quantum harmonic oscillator. The only difference is that \hbar is replaced by $\hbar \xi_k$.

The eigenfunction associated with the lowest eigenvalue of (4.8) is given by

$$\phi_{0k}(x) = \left(\frac{m\omega}{\pi \hbar \xi_k} \right)^{1/4} \exp \left\{ -\frac{m\omega}{2\hbar \xi_k} x^2 \right\}. \quad (4.9)$$

The corresponding eigenfunction of H is therefore given by

$$\begin{aligned}
\phi_0(x) &= \mathbf{e}_1 \phi_{01}(x) + \mathbf{e}_2 \phi_{02}(x) \\
&= \mathbf{e}_1 \left(\frac{m\omega}{\pi \hbar \xi_1} \right)^{1/4} \exp \left\{ -\frac{m\omega}{2\hbar \xi_1} x^2 \right\} + \mathbf{e}_2 \left(\frac{m\omega}{\pi \hbar \xi_2} \right)^{1/4} \exp \left\{ -\frac{m\omega}{2\hbar \xi_2} x^2 \right\} \\
&= \left(\frac{m\omega}{\pi \hbar} \right)^{1/4} \left(\frac{\mathbf{e}_1}{\xi_1^{1/4}} + \frac{\mathbf{e}_2}{\xi_2^{1/4}} \right) \left\{ \mathbf{e}_1 \exp \left[-\frac{m\omega}{2\hbar \xi_1} x^2 \right] + \mathbf{e}_2 \exp \left[-\frac{m\omega}{2\hbar \xi_2} x^2 \right] \right\}. \quad (4.10)
\end{aligned}$$

It can be shown [15] that for any bicomplex number $w = z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2$,

$$\exp \{w\} = \mathbf{e}_1 \exp \{z_1\} + \mathbf{e}_2 \exp \{z_2\}. \quad (4.11)$$

This holds also for any polynomial function $Q(w)$, that is,

$$Q(z_1 \mathbf{e}_1 + z_2 \mathbf{e}_2) = \mathbf{e}_1 Q(z_1) + \mathbf{e}_2 Q(z_2). \quad (4.12)$$

Moreover, if $\xi = \xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2$ with ξ_1 and ξ_2 positive, we have

$$\frac{1}{\xi^{1/4}} = \frac{\mathbf{e}_1}{\xi_1^{1/4}} + \frac{\mathbf{e}_2}{\xi_2^{1/4}}. \quad (4.13)$$

Substituting (4.11) and (4.13) in (4.10), we get

$$\phi_0(x) = \left(\frac{m\omega}{\pi \hbar \xi} \right)^{1/4} \exp \left\{ -\frac{m\omega}{2\hbar \xi} x^2 \right\}. \quad (4.14)$$

From the normalization of ϕ_{01} and ϕ_{02} , we find that

$$\begin{aligned} (\phi_0, \phi_0) &= \int_{-\infty}^{\infty} [\mathbf{e}_1 \bar{\phi}_{01}(x) \phi_{01}(x) + \mathbf{e}_2 \bar{\phi}_{02}(x) \phi_{02}(x)] dx \\ &= \mathbf{e}_1 + \mathbf{e}_2 = 1. \end{aligned} \quad (4.15)$$

The eigenfunction associated with the l th eigenvalue of (4.8) is given by [22]

$$\phi_{lk}(x) = \left[\sqrt{\frac{m\omega}{\pi \hbar \xi_k}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta_k^2/2} H_l(\theta_k), \quad (4.16)$$

where

$$\theta_k = \sqrt{\frac{m\omega}{\hbar \xi_k}} x \quad (4.17)$$

and $H_l(\theta_k)$ is the Hermite polynomial of order l . Just as in (3.34) we now define

$$\phi_l(x) = \mathbf{e}_1 \phi_{l1}(x) + \mathbf{e}_2 \phi_{l2}(x). \quad (4.18)$$

We therefore obtain

$$\begin{aligned} \phi_l(x) &= \mathbf{e}_1 \left[\sqrt{\frac{m\omega}{\pi \hbar \xi_1}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta_1^2/2} H_l(\theta_1) + \mathbf{e}_2 \left[\sqrt{\frac{m\omega}{\pi \hbar \xi_2}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta_2^2/2} H_l(\theta_2) \\ &= \left\{ \mathbf{e}_1 \left[\sqrt{\frac{m\omega}{\pi \hbar \xi_1}} \frac{1}{2^l l!} \right]^{1/2} + \mathbf{e}_2 \left[\sqrt{\frac{m\omega}{\pi \hbar \xi_2}} \frac{1}{2^l l!} \right]^{1/2} \right\} \\ &\quad \cdot \left\{ \mathbf{e}_1 e^{-\theta_1^2/2} + \mathbf{e}_2 e^{-\theta_2^2/2} \right\} \{ \mathbf{e}_1 H_l(\theta_1) + \mathbf{e}_2 H_l(\theta_2) \}. \end{aligned} \quad (4.19)$$

Letting $\theta := \mathbf{e}_1\theta_1 + \mathbf{e}_2\theta_2$ and making use of (4.11)–(4.13), we finally obtain

$$\phi_l(x) = \left[\sqrt{\frac{m\omega}{\pi\hbar\xi}} \frac{1}{2^l l!} \right]^{1/2} e^{-\theta^2/2} H_l(\theta), \quad (4.20)$$

where

$$H_l(\theta) := \mathbf{e}_1 H_l(\theta_1) + \mathbf{e}_2 H_l(\theta_2) \quad (4.21)$$

is a hyperbolic Hermite polynomial of order l .

Equation (4.20) is one of the central results of this paper. It expresses normalized eigenfunctions of the bicomplex harmonic oscillator Hamiltonian purely in terms of hyperbolic constants and functions, with no reference to a particular representation like $\{\mathbf{e}_k\}$. Indeed ξ can be viewed as a \mathbb{D}^+ constant, θ is equal to $\sqrt{m\omega/\hbar\xi}x$ and $H_l(\theta)$ is just the Hermite polynomial in θ .

Let \tilde{M} be the collection of all finite linear combinations of bicomplex functions $\phi_l(x)$, with bicomplex coefficients. That is,

$$\tilde{M} := \left\{ \sum_l w_l \phi_l(x) \mid w_l \in \mathbb{T} \right\}. \quad (4.22)$$

It is easy to see that $\phi_l(x)$ is a function like $u(x)$ defined in (4.2). Thus \tilde{M} is a submodule of the module M_S defined earlier in terms of functions $u(x)$. Moreover, \tilde{M} is isomorphic to the module M defined in section 3.3.

In section 3.3, the most general eigenket of H was written as in (3.47). The corresponding eigenfunction has the form

$$\phi(x) = \mathbf{e}_1 w_{l1} \phi_{l1}(x) + \mathbf{e}_2 w_{l'2} \phi_{l'2}, \quad (4.23)$$

with w_{l1} and $w_{l'2}$ in $\mathbb{C}(i_1)$. The eigenfunction can be written explicitly as

$$\phi(x) = \left[\frac{m\omega}{\pi\hbar} \right]^{1/4} \left\{ \mathbf{e}_1 \frac{w_{l1} e^{-\theta_1^2/2}}{\sqrt{2^l l!} \sqrt{\xi_1}} H_l(\theta_1) + \mathbf{e}_2 \frac{w_{l'2} e^{-\theta_2^2/2}}{\sqrt{2^{l'} (l')!} \sqrt{\xi_2}} H_{l'}(\theta_2) \right\}. \quad (4.24)$$

The function ϕ is normalized, *i.e.* $(\phi, \phi) = 1$, if

$$|w_{l1}|^2 \mathbf{e}_1 + |w_{l'2}|^2 \mathbf{e}_2 = 1. \quad (4.25)$$

This means that $|w_{l1}| = 1 = |w_{l'2}|$. From the properties of real Hermite polynomials, one can show that two functions $\phi(x)$ associated with eigenvalues whose difference is not in the null cone, are orthogonal.

The function $\phi(x)$ can also be expressed in terms of the hyperbolic units 1 and j instead of \mathbf{e}_1 and \mathbf{e}_2 . Letting $w_{l1} = 1 = w_{l'2}$, we get

$$\begin{aligned} \phi(x) = \left[\frac{m\omega}{\pi\hbar} \right]^{1/4} \frac{1}{2} \left\{ \left[\frac{e^{-\theta_1^2/2}}{\sqrt{2^l l! \sqrt{\xi_1}}} H_l(\theta_1) + \frac{e^{-\theta_2^2/2}}{\sqrt{2^{l'} (l')! \sqrt{\xi_2}}} H_{l'}(\theta_2) \right] \right. \\ \left. + j \left[\frac{e^{-\theta_1^2/2}}{\sqrt{2^l l! \sqrt{\xi_1}}} H_l(\theta_1) - \frac{e^{-\theta_2^2/2}}{\sqrt{2^{l'} (l')! \sqrt{\xi_2}}} H_{l'}(\theta_2) \right] \right\}. \end{aligned} \quad (4.26)$$

5 Outlook and conclusion

Bicomplex numbers include complex numbers as a subset, just like hyperbolic numbers include real numbers. It should come as no surprise then that the eigenkets and eigenfunctions of the standard harmonic oscillator Hamiltonian can be recovered from the bicomplex ones. This, in fact, can be done in three different ways, as can be seen for instance from (3.47). First we set $\xi_1 = 1 = \xi_2$. We then let either (i) $w_{l1} = \mathbf{e}_1$ and $w_{l'2} = 0$, or (ii) $w_{l1} = 0$ and $w_{l'2} = \mathbf{e}_2$, or finally (iii) $l = l'$, $w_{l1} = \mathbf{e}_1$ and $w_{l'2} = \mathbf{e}_2$. In each case the eigenkets make up a structure isomorphic to the standard harmonic oscillator eigenvectors.

The usual Hermite polynomials can similarly be recovered from the bicomplex ones. Looking at (4.21), we can see that both $P_1(H_l(\theta))$ and $P_2(H_l(\theta))$ are real-valued Hermite polynomials. So is $H_l(\theta)$ itself, in the special case where $\theta_1 = \theta_2 = \theta$.

In the module \tilde{M} defined in (4.22), the coefficients w_l are bicomplex numbers. If they are restricted to elements of $\mathbb{C}(i_1)$, then the set of linear combinations makes up a vector space \tilde{V} , isomorphic to the space V defined after (3.35). The space \tilde{V} is not restricted to standard Hermite polynomials but contains all the hyperbolic ones.

We should note that the infinite-dimensional modules M , \tilde{M} and M_S that we have introduced have been defined, so to speak, in a minimal way. Indeed they do contain all eigenkets or eigenfunctions of the bicomplex harmonic oscillator Hamiltonian, but they only involve finite linear combinations of elements. This was done purposely, for in this way we are able to avoid convergence problems or other delicate issues involved in complete spaces like $L^2(\mathbb{R})$. Our modules do not have a property of completeness. Indeed Cauchy sequences of elements of \tilde{M} , for instance, do not in general converge to elements of \tilde{M} .

We believe, however, that all our modules can be extended to complete ones, that is, to what can be called infinite-dimensional bicomplex Hilbert spaces. To motivate this, let $|\phi\rangle$ be an arbitrary ket in M . We can define the norm of $|\phi\rangle$ as the modulus of the square root of the scalar product of $|\phi\rangle$ with itself:

$$\text{Norm}(|\phi\rangle) := \left| \sqrt{(|\phi\rangle, |\phi\rangle)} \right|. \quad (5.1)$$

One can show that up to a positive factor, this norm coincides with the square root of the sum of the squares of the natural norms of $P_1(|\phi\rangle)$ and $P_2(|\phi\rangle)$. If we define a Cauchy sequence of elements of M as one for which $\text{Norm}(|\phi^m\rangle - |\phi^n\rangle) \rightarrow 0$ as $m, n \rightarrow \infty$, then one can show that a sequence of elements of M is a Cauchy sequence if and only if both its projections P_1 and P_2 are Cauchy sequences. This suggests that a bicomplex Hilbert space can be built from the two separate Hilbert spaces of its projections. This is presently being investigated.

We have shown that the standard quantum-mechanical harmonic oscillator can be generalized to bicomplex numbers. We have not suggested any specific physical interpretation of the bicomplex eigenkets or eigenfunctions. Indeed much work has to be done to assess the extent to which the postulates of standard quantum mechanics can be extended to the bicomplex number system. It appears, however, that other quantum-mechanical problems can be generalized to bicomplex numbers. Here we specifically have in mind the r^{-1} potential, which also lends itself both to algebraic and differential equation treatments [22]. Presumably, real Laguerre polynomials and complex spherical harmonics can be generalized, respectively, to hyperbolic and bicomplex ones. This is also an area worth being investigated.

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References

- [1] Von Neumann J., *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton) 1955.
- [2] Stueckelberg E. C. G., *Helv. Phys. Acta* **33** (1960) 727.
- [3] Stueckelberg E. C. G. and Guenin M., *Helv. Phys. Acta* **34** (1961) 621.
- [4] Reichenbach H., *Philosophic Foundations of Quantum Mechanics* (University of California Press, Berkeley and Los Angeles) 1944.
- [5] Fuchs C. A., in *Quantum Theory: Reconsideration of Foundations*, edited by Khrennikov A. (Växjö University Press, Växjö) 2002, pp. 463–543; also available as quant-ph/0205039.
- [6] Clifton R., Bub J. and Halvorson H., *Found. Phys.* **33** (2003) 1561.

- [7] Birkhoff G. and von Neumann J., *Ann. Math.* **37** (1936) 823.
- [8] Nash C. G. and Joshi G. C., *Int. J. Theor. Phys.* **31** (1992) 965.
- [9] Adler S. L., *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford University Press, Oxford) 1995.
- [10] Oneto A., *Divulg. Mat.* **10** (2002) 161.
- [11] Albert A. A., *Ann. Math.* **48** (1947) 495.
- [12] Millard A. C., *J. Math. Phys.* **38** (1997) 6230.
- [13] Kocik J., *Int. J. Theor. Phys.* **38** (1999) 2221.
- [14] Rochon D. and Tremblay S., *Adv. Appl. Clifford Alg.* **16** (2006) 135.
- [15] Baley Price G., *An Introduction to Multicomplex Spaces and Functions* (Marcel Dekker, New York) 1991.
- [16] Rochon D. and Shapiro M., *An. Univ. Oradea, Fasc. Mat.* **11** (2004) 71.
- [17] Rochon D. and Tremblay S., *Adv. Appl. Clifford Alg.* **14** (2004) 231.
- [18] Gervais Lavoie R., Marchildon L. and Rochon D., *Finite-dimensional bicomplex Hilbert spaces* (arXiv:1003.1122); forthcoming in *Adv. Appl. Clifford Alg.*
- [19] Heisenberg W., *Z. Phys.* **33** (1925) 879; English translation in *Sources of Quantum Mechanics*, edited by van der Waerden B. L. (Dover, New York) 1968, pp. 261–76.
- [20] Dirac P. A. M., *The Principles of Quantum Mechanics* 4th edition (Clarendon Press, Oxford) 1967.
- [21] Eckart C., *Proc. Nat. Acad. Sci. USA* **12** (1926) 473.
- [22] Marchildon L., *Quantum Mechanics: From Basic Principles to Numerical Methods and Applications* (Springer, Berlin) 2002.